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On the Toda lattice equation with self-consistent sources

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Abstract
The *Toda lattice hierarchy with self-consistent sources* and their Lax representation are derived. We construct a *forward* Darboux transformation (FDT) with arbitrary functions of time and a *generalized* forward Darboux transformation (GFDT) for Toda lattice with self-consistent sources (TLSCS), which can serve as a non-auto-Bäcklund transformation between TLSCS with different degrees of sources. With the help of such DT, we can construct many types of solutions to TLSCS, such as rational solution, solitons, positons, negetons, and soliton–positons, soliton–negatons, positon–negatons etc, and study properties and interactions of these solutions.

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1. Introduction

Recently, soliton equations with self-consistent sources (SESCSs) in several (1+1)- and (2+1)-dimensional *continuous* cases, which have very important applications in many fields, such as hydrodynamics, solid-state physics, plasma physics, etc have been widely studied [1–5]. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves [4]. Various methods have been used to construct their solutions, such as inverse scattering methods [2, 3, 6, 7], Darboux transformation methods [8–11], Hirota bilinear methods [12–14] etc. Comparing with Darboux transformations (DT) for soliton equations, generalized binary Darboux transformations (GBDT) with arbitrary functions of time developed in [8–11] provides a non-auto-Bäcklund transformation between two soliton equations with different degrees of sources, and enable us to find various solutions, such as soliton, positon, negaton, soliton–positon, soliton–negaton, positon–negaton and etc.

However, in contrast with the continuous cases, the integrable discrete systems with self-consistent sources and their physical applications have not been studied yet. We will present a way to construct integrable discrete systems with self-consistent sources based on the constrained flows of the integrable discrete system [15] by considering the latter as the stationary equation of the former. Then we derive a *forward* Darboux transformation (FDT) with an arbitrary function of time for the integrable discrete system with self-consistent sources by improving the method in [8–11]. We use a discrete system, the Toda lattice equation with self-consistent sources (TLSCS), to illustrate our method. The FDT with an arbitrary function of time is generated by using a linear combination of two independent eigenfunctions of the Lax pair with the combination coefficient explicitly depending on time. It serves as a non-auto-Bäcklund transformation between two TLSCS with different degrees of sources. We give a formula for multi-time repetition of such DT. We also construct the *generalized* FDT (GFDT) with arbitrary functions of time. The formula for GFDT is quite similar to the generalized *Wronski* determinant formulae [16] and the generalized *Casorati* determinant formulae [17, 18].

It is well known that the Toda lattice equation possesses rich families of solutions including rational solution, solitons, positons, negatons and soliton–positon, soliton–negaton, positon–negaton. Soliton is a fast decaying pulse-like solution without singularity [21]. It describes a wave propagating in a constant speed. Positon is an oscillating and slowly decaying solution with singularities. It leads to a trivial scattering matrix (called 'supertransparent') when inserted as potentials in the finite-difference Schrödinger equation of the corresponding Lax pair [17]. The interaction between positon and other types of solutions is very interesting. There is no phase shift for others during the course of collision. The negaton is another type of solution with singularities, oscillating, but fast decaying and with non-trivial scattering matrix. Unlike the positon case, there is a phase shift for other types of solutions during the interaction between negaton and others.

In the second part of our paper, we show that TLSCS also have such types of solutions. These solutions can be obtained easily through GFDT. Some of these solutions as well as their analytic properties are presented in our paper. Differences between solutions for TLSCS and solutions for the Toda lattice equation are also studied. In particular, a new feature regarding negaton–positon (negaton–soliton) interaction is analysed.

The paper will be organized as follows. We first review the Toda lattice hierarchy in section 2. In section 3, through the nth-constrained flow for the Toda lattice hierarchy we give the Toda lattice hierarchy with self-consistent sources and its Lax representation. On the basis of the Darboux transformation for the Toda lattice equation, we develop a method to construct the FDT and GFDT with arbitrary functions of time for TLSCS in section 4. In section 5, we construct some solutions for TLSCS by using GFDT with arbitrary functions of time. Some properties of these solutions are analysed therein.

2. Toda lattice hierarchy

We first review the Toda lattice hierarchy. Assume $f = f(n,t)$ for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Define shift operator E and difference operator Δ as follows:

$$(Ef)(n,t) = f(n+1,t),$$
$$(\Delta f)(n,t) = (E-1)f(n,t) = f(n+1,t) - f(n,t).$$

We often denote $E^k f(n,t)$ by $f^{(k)}$. The inverse of E and Δ are defined as

$$(E^{-1}f)(n,t) = f(n-1,t),$$
$$(\Delta^{-1}f)(n,t) = \begin{cases} \sum_{i=0}^{n-1} f(i,t) & n \geqslant 1, \\ 0 & n = 0, \\ -\sum_{i=n}^{-1} f(i,t) & n \leqslant -1. \end{cases}$$

Consider the following discrete isospectral problem [17, 19]

$$L\psi = (v\psi)^{(1)} + p\psi + \psi^{(-1)} = \lambda\psi, \tag{1}$$

where $v = v(n,t)$, $p = p(n,t)$, $\psi = \psi(n,\lambda,t)$. This equation can be rewritten as the following 2×2 matrix eigenvalue problem

$$\Psi^{(-1)} = U(v,p,\lambda)\Psi, \tag{2}$$

where

$$U(v,p,\lambda) := \begin{pmatrix} 0 & 1 \\ -v^{(1)} & \lambda - p \end{pmatrix}, \qquad \Psi := \begin{pmatrix} \psi^{(1)} \\ \psi \end{pmatrix}.$$

First consider the following *stationary zero-curvature equation* for generating function Γ [20],

$$\Gamma^{(-1)}U - U\Gamma = 0, \tag{3}$$

where

$$\Gamma := \sum_{i=0}^{+\infty} \Gamma_i \lambda^{-i} = \sum_{i=0}^{+\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i},$$

and a_i, b_i, c_i are the functions of n and t. The first few of these read

$$\begin{aligned}
&a_0 = \tfrac{1}{2}, && b_0 = 0, && c_0 = 0,\\
&a_1 = 0, && b_1 = -1, && c_1 = v^{(1)},\\
&a_2 = v^{(1)}, && b_2 = -p^{(1)}, && c_2 = pv^{(1)},\\
&a_3 = (p + p^{(1)})v^{(1)}, && b_3 = -(p^{(1)2} + v^{(1)} + v^{(2)}), && c_3 = v^{(1)}(p^2 + v + v^{(1)}).
\end{aligned}$$

In general

$$\Delta a_{i+1} = p^{(1)}\Delta a_i - c_i - b_i^{(1)}v^{(2)}, \qquad b_{i+1} = p^{(1)}b_i - \Delta a_i, \qquad c_i = -v^{(1)}b_i^{(-1)}.$$

Define the modification matrix

$$\Delta_n := \operatorname{diag}(b_{n+1} + \delta, \delta),$$

where δ is an arbitrary constant. Let

$$V_n := (\lambda^n\Gamma)_+ + \Delta_n = \sum_{i=0}^{n} \Gamma_i\lambda^{n-i} + \Delta_n,$$

and

$$-\Psi_{t_n} = V_n\Psi, \qquad (n \geqslant 1). \tag{4}$$

Then the compatibility condition of (2) and (4) gives rise to the following zero-curvature representation of the Toda lattice hierarchy

$$U_{t_n} = UV_n - V_n^{(-1)}U, \qquad (n \geqslant 1). \tag{5}$$

Hamiltonian form for the Toda lattice hierarchy is given by

$$\begin{pmatrix} v \\ p \end{pmatrix}_{t_n} = J \begin{pmatrix} \dfrac{\delta H_n}{\delta v} \\ \dfrac{\delta H_n}{\delta p} \end{pmatrix} = J \begin{pmatrix} a_{n+1}^{(-1)}/v \\ -b_{n+1}^{(-1)} \end{pmatrix}$$

with the density $H_n := -b_{n+2}/(n+1)$ and Hamilton operator

$$J := \begin{pmatrix} 0 & v(E^{-1} - 1) \\ (1 - E)v & 0 \end{pmatrix}.$$

For $n = 1$ in (5), we have the well-known Toda lattice equation

$$v_t = vp^{(-1)} - vp, \qquad p_t = v - v^{(1)}. \tag{6}$$

Set $v := \exp(x^{(-1)} - x)$, $p := x_t$, equations (6) can be represented as

$$x_{tt} = \exp(x^{(-1)} - x) - \exp(x - x^{(1)}). \tag{7}$$

3. The Toda lattice hierarchy with self-consistent sources

First we briefly review the nth-constrained flow for the Toda lattice hierarchy [15] which is defined as the following system:

$$\begin{pmatrix} \frac{\delta H_n}{\delta v} \\ \frac{\delta H_n}{\delta p} \end{pmatrix} + \sum_{j=1}^{N} \begin{pmatrix} \frac{\delta \lambda_j}{\delta v} \\ \frac{\delta \lambda_j}{\delta p} \end{pmatrix} = 0, \tag{8a}$$

$$L\phi_{j+} := (v\phi_{j+})^{(1)} + p\phi_{j+} + \phi_{j+}^{(-1)} = \lambda_j \phi_{j+}, \tag{8b}$$

$$L^*\phi_{j-} := v\phi_{j-}^{(-1)} + p\phi_{j-} + \phi_{j-}^{(1)} = \lambda_j \phi_{j-}, \qquad j = 1, \ldots, N, \tag{8c}$$

where

$$(\delta\lambda_j/\delta v, \delta\lambda_j/\delta p)^T = \left(\phi_{j-}^{(-1)}\phi_{j+}, \phi_{j-}\phi_{j+}\right)^T.$$

Define

$$\tilde{\Gamma}_n = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i} + \sum_{j=1}^{N} \frac{1}{\lambda^n(\lambda - \lambda_j)} \begin{pmatrix} -v^{(1)}\phi_{j-}\phi_{j+}^{(1)} & \phi_{j-}^{(1)}\phi_{j+}^{(1)} \\ -v^{(1)}\phi_{j-}\phi_{j+} & v^{(1)}\phi_{j-}\phi_{j+}^{(1)} \end{pmatrix}.$$

The Lax representation for (8a) is [15]

$$\Psi^{(-1)} = U\Psi, \qquad \mu\Psi = \tilde{\Gamma}_n \Psi \tag{9}$$

Following the idea in the continuous case that by treating the constrained flows of soliton equation as the stationary equations of the soliton equation with self-consistent sources, we define the Toda lattice hierarchy with N self-consistent sources as the following system:

$$\begin{pmatrix} v \\ p \end{pmatrix}_{t_n} = J \begin{bmatrix} \frac{\delta H_n}{\delta v} + \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta v} \\ \frac{\delta H_n}{\delta p} + \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta p} \end{bmatrix} = J \begin{bmatrix} a_{n+1}^{(-1)}/v + \sum_{j=1}^{N} \phi_{j-}^{(-1)}\phi_{j+} \\ -b_{n+1}^{(-1)} + \sum_{j=1}^{N} \phi_{j-}\phi_{j+} \end{bmatrix}, \tag{10a}$$

$$L\phi_{j+} = \lambda_j \phi_{j+}, \tag{10b}$$

$$L^*\phi_{j-} = \lambda_j \phi_{j-}, \qquad j = 1, \ldots, N. \tag{10c}$$

The Lax representation for (10a) can be obtained from the Lax representation (9)

$$\Psi^{(-1)} = U\Psi, \qquad -\Psi_{t_n} = (\lambda^n \tilde{\Gamma}_n + \tilde{\Delta}_n)\Psi, \tag{11}$$

where $\tilde{\Delta}_n = \mathrm{diag}\left(b_{n+1} - \sum_{j=1}^{N} \phi_{j-}^{(1)}\phi_{j+}^{(1)} + \delta, \delta\right)$, with δ an arbitrary constant.

For $n = 1$, we get the Toda lattice equation with N self-consistent sources (TLSCS)

$$v_t = v\left(p^{(-1)} + \sum_{j=1}^{N} \phi_{j-}^{(-1)}\phi_{j+}^{(-1)}\right) - v\left(p + \sum_{j=1}^{N} \phi_{j-}\phi_{j+}\right), \tag{12a}$$

$$p_t = v\left(1 + \sum_{j=1}^{N} \phi_{j-}^{(-1)}\phi_{j+}\right) - v^{(1)}\left(1 + \sum_{j=1}^{N} \phi_{j-}\phi_{j+}^{(1)}\right), \tag{12b}$$

$$L\phi_{j+} = \lambda_j \phi_{j+}, \tag{12c}$$

$$L^*\phi_{j-} = \lambda_j \phi_{j-}, \qquad j = 1, \ldots, N. \tag{12d}$$

The Lax representation for TLSCS is obtained from (11) by taking $n = 1$ and $\delta = \lambda/2$. We prefer to rewrite it equivalently in the following scalar form:

$$L\psi = v^{(1)}\psi^{(1)} + p\psi + \psi^{(-1)} = \lambda\psi, \tag{13a}$$

$$-\psi_t = v^{(1)}\psi^{(1)} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} v^{(1)}\phi_{j-}\left(\phi_{j+}^{(1)}\psi - \phi_{j+}\psi^{(1)}\right), \tag{13b}$$

$$L\phi_{j+} = \lambda_j\phi_{j+}, \tag{13c}$$

$$L^*\phi_{j-} = \lambda_j\phi_{j-}, \qquad j = 1, \ldots, N. \tag{13d}$$

With the substitutions

$$v := \exp(x^{(-1)} - x), \qquad p := x_t - \sum_{j=1}^{N} \phi_{j+}\phi_{j-}, \tag{14}$$

equations (12a) and (12b) can be written as

$$x_{tt} = \exp(x^{(-1)} - x)\left(1 + \sum_{j=1}^{N}\phi_{j+}\phi_{j-}^{(-1)}\right) - \exp(x - x^{(1)})\left(1 + \sum_{j=1}^{N}\phi_{j+}^{(1)}\phi_{j-}\right) + \sum_{j=1}^{N}(\phi_{j+}\phi_{j-})_t. \tag{15}$$

4. The Darboux transformations for TLSCS

4.1. The forward DT with arbitrary functions of time

On the basis of the Darboux transformation for the Toda equation in [17, 19], we can find the following theorem.

Theorem 1. *Given the solution $v, p, x, \phi_{i\pm}$ $(i = 1, \ldots, N)$ for (12) and (15), and eigenfunction ψ for (13a) and (13b), let f and g be two independent eigenfunctions of (13a) and (13b) with $\lambda = \mu$. Denote $h = f + \alpha(t)g$ with the coefficient $\alpha(t)$ being an arbitrary differentiable function of t. Then the FDT is defined as follows:*

$$\psi[1] = \psi - \frac{h}{h^{(1)}}\psi^{(1)}, \tag{16a}$$

$$v[1] = v^{(1)}\frac{h^{(1)}h^{(-1)}}{h^2}, \tag{16b}$$

$$p[1] = p - \frac{h}{h^{(1)}} + \frac{h^{(-1)}}{h}, \tag{16c}$$

$$x[1] = x^{(1)} + \frac{1}{2}\log\left(\frac{h}{h^{(1)}}\right)^2, \tag{16d}$$

$$\phi_{i+}[1] = \phi_{i+} - \frac{h}{h^{(1)}}\phi_{i+}^{(1)}, \tag{16e}$$

$$\phi_{i-}[1] = \frac{\Delta^{-1}E(h\phi_{i-}) + \kappa_i(t)}{h}, \qquad i = 1, \ldots, N, \tag{16f}$$

$$\phi_{N+1,+}[1] = c(t)\left(f - \frac{h}{h^{(1)}} f^{(1)}\right), \tag{16g}$$

$$\phi_{N+1,-}[1] = d(t)\frac{1}{h}, \tag{16h}$$

where $c(t)$ and $d(t)$ are the arbitrary fixed differentiable functions of t satisfying $c(t) \cdot d(t) = -\dot{\alpha}/\alpha$, $\kappa_i(t) = \frac{1}{\mu-\lambda_i}\big[v^{(1)}h^{(1)}\phi_{i-} - h\phi_{i-}^{(1)}\big]\big|_{n=0}$. Namely, $v[1], p[1], x[1], \phi_{i\pm}[1]$ $(i = 1, \ldots, N+1)$, $\lambda_{N+1} = \mu$ and $\psi[1]$ gives a new solution to (12) or (15) and (13) with $N+1$ self-consistent sources.

This theorem can be proved by straightforward calculation.

Remark 1. Theorem 1 serves as a non-auto-Bäcklund transformation between two TLSCSs with degrees of sources N and $N+1$

We give an example for obtaining solution via theorem 1.

Example 1 (rational solution)**.** Starting from trivial solution $v = 1$, $p = 0$ and vanishing sources, the Lax pair (13) becomes

$$L\psi = \psi^{(1)} + \psi^{(-1)} = \lambda\psi, \tag{17a}$$

$$-\psi_t = \psi^{(1)}. \tag{17b}$$

Let $\psi = \exp(kn - \mathrm{e}^k t)$ be the solution of (17) w.r.t. $\lambda = 2\cosh(k)$, $f = (an - at + b)\exp(-t)$, $g = \exp(-t)$ be independent solutions of (17) w.r.t. $\lambda = 2$, where $a \in \mathbb{R} - \{0\}$ and $b \in \mathbb{R}$ are arbitrary constants. Let $h = f + \alpha(t)g$ with the differentiable function $\alpha(t)$. Then we obtain the rational solution with a pair of non-vanishing sources

$$\psi_+[1] = \left[1 - \mathrm{e}^k \frac{an + b - at + \alpha(t)}{an + a + b - at + \alpha(t)}\right]\exp(kn - \mathrm{e}^k t),$$

$$v[1] = \frac{[an + a + b - at + \alpha(t)][an - a + b - at + \alpha(t)]}{[an + b - at + \alpha(t)]^2},$$

$$p[1] = \frac{an - a + b - at + \alpha(t)}{an + b - at + \alpha(t)} - \frac{an + b - at + \alpha(t)}{an + a + b - at + \alpha(t)},$$

$$x[1] = \frac{1}{2}\log\left[\frac{an + b - at + \alpha(t)}{an + a + b - at + \alpha(t)}\right]^2,$$

$$\phi_{1+}[1] = -c(t)\frac{a\,\mathrm{e}^{-t}\alpha(t)}{an + a + b - at + \alpha(t)},$$

$$\phi_{1-}[1] = d(t)\frac{\mathrm{e}^t}{an + b - at + \alpha(t)},$$

where $c(t)$ and $d(t)$ satisfying $c(t)\,d(t) = -\dot{\alpha}/\alpha$.

4.2. The multi-time repeated FDT

Theorem 2 (the multi-time repeated FDT)**.** *Given the solution $v, p, x, \phi_{i\pm}$ $(i = 1, \ldots, N)$ for (12) and (15), and eigenfunction ψ for (13a) and (13b), let f_j and g_j be the independent eigenfunctions of (13a) and (13b) w.r.t. distinct μ_j $(j = 1, \ldots, l)$. Let $\alpha_j(t)$ $(j = 1, \ldots, l)$*

be arbitrary smooth functions of t. Denote $h_j = f_j + \alpha_j g_j$. Then the l-times repeated FDT is given as

$$\psi[l] = \frac{\mathrm{cas}(\psi, h_1, \ldots, h_l)}{\mathrm{cas}(h_1, \ldots, h_l)^{(1)}}, \tag{18a}$$

$$v[l] = v^{(l)} \frac{\mathrm{cas}(h_1, \ldots, h_l)^{(1)}\,\mathrm{cas}(h_1, \ldots, h_l)^{(-1)}}{\mathrm{cas}(h_1, \ldots, h_l)^2}, \tag{18b}$$

$$p[l] = p + \frac{\widetilde{\mathrm{cas}}(h_1, \ldots, h_l)^{(-1)}}{\mathrm{cas}(h_1, \ldots, h_l)} - \frac{\widetilde{\mathrm{cas}}(h_1, \ldots, h_l)}{\mathrm{cas}(h_1, \ldots, h_l)^{(1)}}, \tag{18c}$$

$$x[l] = x^{(l)} + \frac{1}{2}\log\left[\frac{\mathrm{cas}(h_1, \ldots, h_l)}{\mathrm{cas}(h_1, \ldots, h_l)^{(1)}}\right]^2, \tag{18d}$$

$$\phi_{i+}[l] = \frac{\mathrm{cas}(\phi_{i+}, h_1, \ldots, h_l)}{\mathrm{cas}(h_1, \ldots, h_l)^{(1)}}, \tag{18e}$$

$$\phi_{i-}[l] = \frac{\overline{\mathrm{cas}}(\phi_{i-}, h_1, \ldots, h_l)}{\mathrm{cas}(h_1, \ldots, h_l)}, \qquad i = 1, \ldots, N, \tag{18f}$$

$$\phi_{N+j,+}[l] = c_j(t)\frac{\mathrm{cas}(f_j, h_1, \ldots, h_l)}{\mathrm{cas}(h_1, \ldots, h_l)^{(1)}}, \tag{18g}$$

$$\phi_{N+j,-}[l] = d_j(t)\frac{\mathrm{cas}(h_1, \ldots, \widehat{h}_j, \ldots, h_l)^{(1)}}{\mathrm{cas}(h_1, \ldots, h_l)}, \qquad j = 1, \ldots, l, \tag{18h}$$

where

$$\mathrm{cas}(h_1, \ldots, h_l) := \det\left(h_i^{(j)}\right)_{\substack{i=1,\ldots,l \\ j=0,\ldots,l-1}},$$

is the Casorati determinant, and

$$\widetilde{\mathrm{cas}}(h_1, \ldots, h_l) := \det\left(h_i^{(j)}\right)_{\substack{i=1,\ldots,l \\ j=0,2,\ldots,l}},$$

$$\overline{\mathrm{cas}}(\phi_{i-}, h_1, \ldots, h_l) := \det\begin{bmatrix} S_i(h_1) & \cdots & S_i(h_l) \\ h_1^{(1)} & \cdots & h_l^{(1)} \\ \vdots & \vdots & \vdots \\ h_1^{(l-1)} & \cdots & h_l^{(l-1)} \end{bmatrix},$$

where

$$S_i(h_j) := S(\phi_{i-}, h_j) = \Delta^{-1}E(\phi_{i-}h_j) + (\mu_j - \lambda_i)^{-1}\left[v^{(1)}h_j^{(1)}\phi_{i-} - h_j\phi_{i-}^{(1)}\right]\Big|_{n=0}.$$

The $\widehat{h}_j$ means the removal of this term and $c_j(t) \cdot d_j(t) = (-1)^j\dot{\alpha}_j/\alpha_j$. Functions $\psi[l], v[l], p[l], x[l], \phi_{i\pm}[l]$ $(i = 1, \ldots, N + l)$ and $\lambda_{N+j} = \mu_j$ $(j = 1, \ldots, l)$ satisfy (13), (12) and (15) with N replaced by $N + l$.

Sketch of proof. For the proof of formulae (18*a*), (18*b*), (18*c*), (18*e*) and (18*g*), see [19]. The formula (18*d*) is proved by using (14). Formulae (18*f*) and (18*h*) are proved by induction. The calculation is lengthy but rather straight forward. We omit it. □

Remark 2. The multi-time repeated FDT (18) provides a non-auto-Bäcklund transformation between two TLSCSs with degrees N and $N + l$.

4.3. The generalized forward Darboux transformations

It is well known from [17] that the positon solution of the Toda lattice is obtained by computing the limit $k_2 \to k_1$ in the result of two-step DT, where $k_{1,2}$ are the parameters of eigenfunction with which DT is generated. In order to construct positon solutions for TLSCS, similar consideration can be made in our case of FDT with arbitrary functions of time. However, in our case, the arbitrary time functions $\alpha_j(t)$ must be carefully chosen to balance the divergence of sources. Possible candidates for $\alpha_j(t)$ are exponential functions. By using it, we get the following GFDT.

Theorem 3 (GFDT)**.** *Given the solution $v, p, x, \phi_{i\pm}$ $(i = 1, \ldots, N)$ for (12) and (15), and eigenfunction ψ for (13a) and (13b), let F_j, G_j $(j = 1, \ldots, l)$ be pairs of independent eigenfunctions of (13a) and (13b) corresponding to distinct λ_{N+j}. Let f_r, g_r $(r = 1, \ldots, I)$ be pairs of independent eigenfunctions of (13a) and (13b) corresponding to $\lambda_{N+l+r} = \lambda_{N+l+r}(\omega_r)$, where $\lambda_{N+l+r}(\omega_r)$ is an analytic function of the parameter $\omega_r \in \mathbb{C}$. Let $\alpha_j(t)$ $(j = 1, \ldots, l)$, $\beta_r(t) (r = 1, \ldots, I)$ be arbitrary differentiable functions of t. Let $m_r \in \mathbb{N}, m_r \geqslant 2\, (r = 1, \ldots, I)$, $\mathbf{m} := (m_1, \ldots, m_I)$. Denote $q_j = F_j + \alpha_j(t) G_j$, $h_r = f_r + g_r$ respectively. Then the following GFDT with l times of FDT with arbitrary time functions $\alpha_j(t)$ and I times of generalized FDT of multiplicities m_r with arbitrary time functions $\beta_r(t)$ is given by*

$$\psi[l, \mathbf{m}] = \frac{\Delta_{l,\mathbf{m}}(\psi)}{\Delta_{l,\mathbf{m}}^{(1)}}, \tag{19a}$$

$$v[l, \mathbf{m}] = v^{(l+|\mathbf{m}|)} \frac{\Delta_{l,\mathbf{m}}^{(1)} \Delta_{l,\mathbf{m}}^{(-1)}}{\Delta_{l,\mathbf{m}}^{2}}, \tag{19b}$$

$$p[l, \mathbf{m}] = p + \frac{\widetilde{\Delta}_{l,\mathbf{m}}^{(-1)}}{\Delta_{l,\mathbf{m}}} - \frac{\widetilde{\Delta}_{l,\mathbf{m}}}{\Delta_{l,\mathbf{m}}^{(1)}}, \tag{19c}$$

$$x[l, \mathbf{m}] = x^{(l+|\mathbf{m}|)} + \frac{1}{2} \log \left(\frac{\Delta_{l,\mathbf{m}}}{\Delta_{l,\mathbf{m}}^{(1)}} \right)^2, \tag{19d}$$

$$\phi_{i+}[l, \mathbf{m}] = \frac{\Delta_{l,\mathbf{m}}(\phi_{i+})}{\Delta_{l,\mathbf{m}}^{(1)}}, \tag{19e}$$

$$\phi_{i-}[l, \mathbf{m}] = \frac{\overline{\Delta}_{l,\mathbf{m}}(\phi_{i-})}{\Delta_{l,m}}, \qquad i = 1, \ldots, N, \tag{19f}$$

$$\phi_{N+j,+}[l, \mathbf{m}] = c_j(t) \frac{\Delta_{l,\mathbf{m}}(F_j)}{\Delta_{l,\mathbf{m}}^{(1)}}, \tag{19g}$$

$$\phi_{N+j,-}[l, \mathbf{m}] = d_j(t) \frac{\Delta_{l,\mathbf{m}}^{j\ \ (1)}}{\Delta_{l,\mathbf{m}}}, \qquad j = 1, \ldots, l, \tag{19h}$$

$$\phi_{N+l+r,+}[l, \mathbf{m}] = -\frac{\Delta_{l,\mathbf{m}}(f_r)}{\Delta_{l,\mathbf{m}}^{(1)}}, \tag{19i}$$

$$\phi_{N+l+r,-}[l, \mathbf{m}] = \dot{\beta}_r \frac{\Delta_{l,\mathbf{m}}^{l+r\ (1)}}{\Delta_{l,\mathbf{m}}}, \qquad r = 1, \ldots, I, \tag{19j}$$

where $c_j(t)\cdot d_j(t) = (-1)^j\dot{\alpha}_j/\alpha_j$, $|\mathbf{m}| := \sum m_r$. *Symbols are defined as follows:*

$$\begin{aligned}\Delta_{l,\mathbf{m}}(\psi) := \operatorname{cas}\big(&\psi, q_1, q_2, \ldots, q_l;\\ &h_1, \partial_{\omega_1}h_1, \partial^2_{\omega_1}h_1, \ldots, \partial^{m_1-1}_{\omega_1}h_1 + (-1)^{m_1-1}\beta_1(t)g_1; \cdots;\\ &h_I, \partial_{\omega_I}h_I, \partial^2_{\omega_I}h_I, \ldots, \partial^{m_I-1}_{\omega_I}h_I + (-1)^{m_I-1}\beta_I(t)g_I\big),\end{aligned}$$

$$\begin{aligned}\Delta_{l,\mathbf{m}} := \operatorname{cas}\big(&q_1, q_2, \ldots, q_l;\\ &h_1, \partial_{\omega_1}h_1, \partial^2_{\omega_1}h_1, \ldots, \partial^{m_1-1}_{\omega_1}h_1 + (-1)^{m_1-1}\beta_1(t)g_1; \cdots;\\ &h_I, \partial_{\omega_I}h_I, \partial^2_{\omega_I}h_I, \ldots, \partial^{m_I-1}_{\omega_I}h_I + (-1)^{m_I-1}\beta_I(t)g_I\big),\end{aligned}$$

$$\begin{aligned}\widetilde{\Delta}_{l,\mathbf{m}} := \widetilde{\operatorname{cas}}\big(&q_1, q_2, \ldots, q_l;\\ &h_1, \partial_{\omega_1}h_1, \partial^2_{\omega_1}h_1, \ldots, \partial^{m_1-1}_{\omega_1}h_1 + (-1)^{m_1-1}\beta_1(t)g_1; \cdots;\\ &h_I, \partial_{\omega_I}h_I, \partial^2_{\omega_I}h_I, \ldots, \partial^{m_I-1}_{\omega_I}h_I + (-1)^{m_I-1}\beta_I(t)g_I\big),\end{aligned}$$

$$\begin{aligned}\Delta^j_{l,\mathbf{m}} := \operatorname{cas}\big(&q_1, \ldots, \widehat{q_j}, \ldots, q_l;\\ &h_1, \partial_{\omega_1}h_1, \partial^2_{\omega_1}h_1, \ldots, \partial^{m_1-1}_{\omega_1}h_1 + (-1)^{m_1-1}\beta_1(t)g_1; \cdots;\\ &h_I, \partial_{\omega_I}h_I, \partial^2_{\omega_I}h_I, \ldots, \partial^{m_I-1}_{\omega_I}h_I + (-1)^{m_I-1}\beta_I(t)g_I\big) \qquad (1 \leqslant j \leqslant l),\end{aligned}$$

$$\begin{aligned}\Delta^{l+r}_{l,\mathbf{m}} := \operatorname{cas}\big(&q_1, q_2, \ldots, q_l;\\ &h_1, \partial_{\omega_1}h_1, \partial^2_{\omega_1}h_1, \ldots, \partial^{m_1}_{\omega_1}h_1 + (-1)^{m_1-1}\beta_1(t)g_1; \cdots;\\ &h_r, \partial_{\omega_r}h_r, \partial^2_{\omega_r}h_r, \ldots, \partial^{m_r-2}_{\omega_r}h_r; \cdots;\\ &h_I, \partial_{\omega_I}h_I, \partial^2_{\omega_I}h_I, \ldots, \partial^{m_I-1}_{\omega_I}h_I + (-1)^{m_I-1}\beta_I(t)g_I\big) \qquad (1 \leqslant r \leqslant I),\end{aligned}$$

where cas *and* $\widetilde{\text{cas}}$ *are defined in theorem 2.*

$$\begin{aligned}\overline{\Delta}_{l,\mathbf{m}}(\phi_{i-}) := \det\big(&\mathbf{w}_i(q_1), \ldots, \mathbf{w}_i(q_1);\\ &\mathbf{w}_i(h_1), \partial_{\omega_1}\mathbf{w}_i(h_1), \partial^2_{\omega_1}\mathbf{w}_i(h_1), \ldots, \partial^{m_1-1}_{\omega_1}\mathbf{w}_i(h_1) + (-1)^{m_1-1}\beta_1\mathbf{w}_i(g_1); \cdots;\\ &\mathbf{w}_i(h_I), \partial_{\omega_I}\mathbf{w}_i(h_I), \partial^2_{\omega_I}\mathbf{w}_i(h_I), \ldots, \partial^{m_I-1}_{\omega_I}\mathbf{w}_i(h_I) + (-1)^{m_I-1}\beta_I\mathbf{w}_i(g_I)\big),\end{aligned}$$

where for any solution ψ *of (13a) with eigenvalue* λ, $\mathbf{w}_i(\psi)$ *is* $(l + |\mathbf{m}|)$*-dimensional column vector defined as*

$$\mathbf{w}_i(\psi) = (S_i(\psi), \psi^{(1)}, \ldots, \psi^{(l+|\mathbf{m}|-1)})^T,$$

and $S_i(\psi)$ *is a scalar defined as*

$$S_i(\psi) := S(\phi_{i-}, \psi) = \Delta^{-1}E(\phi_{i-}\psi) + (\lambda - \lambda_i)^{-1}[v^{(1)}\psi^{(1)}\phi_{i-} - \phi_{i-}\psi^{(1)}]|_{n=0}.$$

Then $\psi[l, \mathbf{m}]$, $v[l, \mathbf{m}]$, $p[l, \mathbf{m}]$, $x[l, \mathbf{m}]$, $\phi_{i\pm}[l, \mathbf{m}]$ *and* $\lambda_i (i = 1, \ldots, N + l + I)$ *satisfy (13) and (12) and (15) with N replaced by* $N + l + I$.

Proof. Without loss of generality, we only prove the special case $l = 0, I = 1$. The multiplicity m_1 is denoted by m ($m \geqslant 2$). And $\lambda_1(\omega_1)$ is denoted by $\lambda(\omega)$, which is analytic function of parameter ω. f_1, g_1, h_1 and $\beta_1(t)$ are denoted by f, g, h and $\beta(t)$, respectively.

Let $\omega_s = \omega + \varepsilon e_s$, where e_s $(s = 1, \ldots, m)$ are distinct complex constants, ε is a small parameter. Define $\varrho_s(t) = \exp(\Omega_s b_s(t))$, where $b_s(t)$ are the arbitrary differentiable functions of t satisfying $\sum_{s=1}^m b_s(t) = \beta(t)$, and

$$\Omega_s = \frac{1}{(m-1)!}\prod_{\substack{1\leqslant i\leqslant m\\ i\neq s}}(\omega_i - \omega_s), \qquad p_s = \frac{1}{(m-1)!}\prod_{\substack{1\leqslant i\leqslant m\\ i\neq s}}(e_i - e_s).$$

We have the obvious relation $\Omega_s = \varepsilon^{m-1}p_s$.

We have the following important observations. Let $\mathbf{u}(\lambda)$, $\mathbf{v}(\lambda)$ be two column vectors of dimension m, $\bar{\mathbf{u}}(\lambda)$, $\bar{\mathbf{v}}(\lambda)$ be two column vectors of dimension $m-1$, whose components are analytic functions of $\lambda = \lambda(\omega)$. Denote $\mathbf{u}_s = \mathbf{u}(\lambda(\omega_s))$, $\mathbf{v}_s = \mathbf{v}(\lambda(\omega_s))$, $\bar{\mathbf{u}}_s = \bar{\mathbf{u}}(\lambda(\omega_s))$, $\bar{\mathbf{v}}_s = \bar{\mathbf{v}}(\lambda(\omega_s))$ then

$$\begin{aligned} &\det(\mathbf{u}_1 + \varrho_1\mathbf{v}_1, \ldots, \mathbf{u}_m + \varrho_m\mathbf{v}_m) \\ &\quad = \frac{\varepsilon^{\frac{1}{2}m(m-1)}}{1!\cdots(m-1)!} \prod_{1\leqslant i<j\leqslant m} (e_j - e_i) \det\big(\mathbf{u}+\mathbf{v}, \partial_\omega(\mathbf{u}+\mathbf{v}), \ldots, \partial_\omega^{m-1}(\mathbf{u}+\mathbf{v}) \\ &\qquad + (-1)^{m-1}\beta(t)\mathbf{v}\big) + o\big(\varepsilon^{\frac{1}{2}m(m-1)}\big), \end{aligned} \tag{20}$$

$$\begin{aligned} &\det(\bar{\mathbf{u}}_1 + \varrho_1\bar{\mathbf{v}}_1, \ldots, \widehat{\bar{\mathbf{u}}_r + \varrho_r\bar{\mathbf{v}}_r}, \ldots, \bar{\mathbf{u}}_m + \varrho_m\bar{\mathbf{v}}_m) = \frac{\varepsilon^{\frac{1}{2}(m-2)(m-1)}}{1!\ldots(m-2)!} \\ &\qquad \times \prod_{\substack{1\leqslant i<j\leqslant m\\ i,j\neq r}} (e_j - e_i) \det\big(\bar{\mathbf{u}}+\bar{\mathbf{v}}, \partial_\omega(\bar{\mathbf{u}}+\bar{\mathbf{v}}), \ldots, \partial_\omega^{m-2}(\bar{\mathbf{u}}+\bar{\mathbf{v}})\big) + o\big(\varepsilon^{\frac{1}{2}(m-2)(m-1)}\big). \end{aligned} \tag{21}$$

These observations can easily be obtained by inserting the Taylor expansion of $\mathbf{u}_s$, $\mathbf{v}_s$ and $\bar{\mathbf{u}}_s$, $\bar{\mathbf{v}}_s$ at $\varepsilon = 0$ into the determinant.

By applying multi-time repeated FDT (theorem 2) with $h(\lambda(\omega_s)) = f(\lambda(\omega_s)) + \varrho_s(t)g(\lambda(\omega_s))$, $c_s = -1$, $d_s = (-1)^{s-1}\dot{\varrho}_s/\varrho_s (s = 1, \ldots, m)$, we obtain $\psi[m]$, $v[m]$, $p[m]$, $x[m]$, $\phi_{i\pm}[m]$ $(i = 1, \ldots, N+m)$ and $\lambda_{N+s} = \lambda(\omega_s)$ satisfying (13), (12) and (15) with $N+m$ self-consistent sources. Since such solution contain an arbitrary parameter ε, letting $\varepsilon \to 0$ will also give rise to solution to (13), (12) and (15) with $N+m$ self-consistent sources. So by doing this with use of (20) and (21), we get the special case ($l = 0$, $\mathbf{m} = m$) of formulae (19a), (19b), (19c), (19d), (19e) and (19f). We also obtain m self-consistent sources

$$\phi_{N+s,+}[m] = -\frac{\operatorname{cas}\big(f, h, \ldots, \partial_\omega^{m-1}h\big)}{\operatorname{cas}\big(h, \ldots, \partial_\omega^{m-1}h + (-1)^{m-1}\beta(t)g\big)},$$

$$\phi_{N+s,-}[m] = \dot{b}_s \frac{\operatorname{cas}\big(h, \ldots, \partial_\omega^{m-2}h\big)}{\operatorname{cas}\big(h, \ldots, \partial_\omega^{m-1}h + (-1)^{m-1}\beta(t)g\big)}, \qquad s = 1, \ldots, m.$$

Since $\phi_{N+s,+}[m]$ are equal, $\phi_{N+s,-}[m]$ differ only in coefficients $\dot{b}_s$ and they are all eigenfunctions w.r.t. eigenvalue $\lambda(\omega)$, it is reasonable to combine m self-consistent sources to one self-consistent source by denoting

$$\phi_{N+1,+}[0, m] = \phi_{N+1,+}[m],$$

$$\phi_{N+1,-}[0, m] = \sum_{s=1}^{m} \phi_{N+s,-}[m].$$

Thus we arrive at (19i) and (19j).

The general case can be proved by repeating above procedure. □

5. Solutions of TLSCS

The GFDT technique enables us to construct various types of solutions to the TLSCS. Starting from trivial solution $v = 1$, $p = 0$ for (12) with $N = 0$, by choosing specific solutions of Lax pair (17) and specific l, I and $\mathbf{m}$ in theorem 3, we can construct multi-solitons solutions, (multi-)positon solutions (or higher order), (multi-)negaton solutions (or higher order) and (multi-)soliton–positon solutions, (multi-)soliton–negaton solutions, (multi-)positon–negaton solutions etc.

5.1. Soliton solutions

Let $F = \nu^n \exp(n\gamma - \nu\,\mathrm{e}^{\gamma} t)$, $G = \nu^n \exp(-n\gamma - \nu\,\mathrm{e}^{-\gamma} t)$ be solutions of (17) with respect to $\lambda = 2\nu\cosh(\gamma)$, where $\gamma \in \mathbb{R}\backslash\{0\}$. Let $\alpha = \exp(-2a(t))$, where $a(t)$ is an arbitrary differentiable functions of t, $\nu = \pm 1$. Define

$$q = F + \alpha G = 2\nu^n \exp(-\nu\cosh(\gamma)t - a)\cosh(U), \qquad U := n\gamma - \nu\sinh(\gamma)t + a.$$

Then using theorem 3 with $l = 1$, $I = 0$, we have the 1-soliton solution for (12) or (15) with $N = 1$.

$$\psi^{\mathrm{sol}} = \exp(\mathrm{i}kn - \mathrm{e}^{\mathrm{i}k}t)\left[1 - \nu\,\mathrm{e}^{\mathrm{i}k}\frac{\cosh(U)}{\cosh(U+\gamma)}\right],$$

$$v^{\mathrm{sol}} = \frac{\cosh(U-\gamma)\cosh(U+\gamma)}{\cosh^2(U)},$$

$$p^{\mathrm{sol}} = \nu\frac{\cosh(U-\gamma)}{\cosh(U)} - \nu\frac{\cosh(U)}{\cosh(U+\gamma)},$$

$$x^{\mathrm{sol}} = \log\left[\frac{\cosh(U)}{\cosh(U+\gamma)}\right],$$

$$\phi_+^{\mathrm{sol}} = -\nu^n\sinh(\gamma)c(t)\frac{\exp(-\nu\cosh(\gamma)t - a)}{\cosh(U+\gamma)},$$

$$\phi_-^{\mathrm{sol}} = \nu^n d(t)\frac{\exp(\nu\cosh(\gamma)t + a)}{2\cosh(U)},$$

where $c(t)d(t) = 2\dot{a}$.

The 1-soliton solution of TLSCS is similar to the 1-soliton of ordinary Toda lattice equation [21]. The main difference between them is that the travelling speed of 1-soliton solution for TLSCS can vary with t, for the travelling speed $\frac{\nu\sinh\gamma - \dot{a}}{\gamma}$. The multi-soliton solutions for TLSCS (12) can be constructed similarly. We omit it.

5.2. Positon solutions and negaton solutions

5.2.1. One-positon solution. Let $f = c_1\exp(-t\cos\omega)\cos(n\omega - t\sin\omega)$, $g = c_2\exp(-t\cos\omega)\sin(n\omega - t\sin\omega)$ be solutions of (17) w.r.t. $\lambda = 2\cos\omega$. Set constants $c_1 = \cos\theta$, $c_2 = \sin\theta$, where $\theta \neq \frac{n}{2}\pi$ $(n \in \mathbb{Z})$. Define

$$h = f + g = \exp(-t\cos\omega)\cos(Y), \qquad Y = n\omega - t\sin\omega - \theta,$$

and

$$Z = \partial_\omega Y = n - t\cos\omega, \qquad \eta = Z + \tfrac{1}{2}\beta(t)\sin(2\theta),$$

where $\beta(t)$ is an arbitrary differentiable function. Then using theorem 3 with the specification $l = 0$, $I = 1$, $m_1 = 2$, one gets one-positon solution for TLSCS (12) with $N = 1$.

$$v^{\mathrm{pos}} = \frac{\left[\left(\eta + \frac{3}{2}\right)\sin\omega + \frac{1}{2}\sin(2Y + 3\omega)\right]\left[\left(\eta - \frac{1}{2}\right)\sin\omega + \frac{1}{2}\sin(2Y - \omega)\right]}{\left[\left(\eta + \frac{1}{2}\right)\sin\omega + \frac{1}{2}\sin(2Y + \omega)\right]^2}, \tag{22a}$$

$$p^{\mathrm{pos}} = \frac{\eta\sin(2\omega) + \sin(2Y)}{\left(\eta + \frac{1}{2}\right)\sin\omega + \frac{1}{2}\sin(2Y + \omega)} - \frac{(\eta + 1)\sin(2\omega) + \sin(2Y + 2\omega)}{\left(\eta + \frac{3}{2}\right)\sin\omega + \frac{1}{2}\sin(2Y + 3\omega)}, \tag{22b}$$

$$x^{\mathrm{pos}} = \frac{1}{2}\log\left[\frac{\left(\eta + \frac{1}{2}\right)\sin\omega + \frac{1}{2}\sin(2Y + \omega)}{\left(\eta + \frac{3}{2}\right)\sin\omega + \frac{1}{2}\sin(2Y + 3\omega)}\right]^2, \tag{22c}$$

$$\phi_{+}^{\mathrm{pos}} = -\sin(2\theta)\frac{\exp(-t\cos\omega)\sin^2\omega\cos(Y+\omega)}{\left(\eta+\frac{3}{2}\right)\sin\omega+\frac{1}{2}\sin(2Y+3\omega)}, \tag{22d}$$

$$\phi_{-}^{\mathrm{pos}} = -\dot{\beta}\frac{\exp(t\cos\omega)\cos(Y+\omega)}{\left(\eta+\frac{1}{2}\right)\sin\omega+\frac{1}{2}\sin(2Y+\omega)}. \tag{22e}$$

The scattering properties can be analysed, resembling with [17]. It is noted that the scattering matrix is an identity matrix, which is the basic feature for positon solutions.

Then we discuss the analytic properties of one-positon solution. An investigation on positon profile x^{pos} shows that the one-positon profile decays to zero slowly at $\pm\infty$, oscillates and possesses singularities during their propagation. The singularities of one positon profile are determined by zeros of

$$\Delta^{\mathrm{pos}} := \left(\eta+\tfrac{1}{2}\right)\sin\omega+\tfrac{1}{2}\sin(2Y+\omega).$$

The speed of the positon profile are defined by the speed of singularities [16]. Assuming $n_0(t)$ to be one of the places where singularity occurs, differentiating by t on the two side of the equation $\Delta^{\mathrm{pos}}(n_0, t) = 0$, one finds that the singularity propagation is governed by a nonlinear ODE

$$\dot{n}_0 = \frac{\sin\omega}{\omega}\left[1+\frac{\omega\cos\omega-\frac{1}{2}\omega\dot{\beta}\sin(2\theta)-\sin\omega}{\sin\omega+\omega\cos(2n_0\omega-2t\sin\omega-2\theta+\omega)}\right].$$

The singularities of ϕ_{+}^{pos} and ϕ_{-}^{pos} are determined by zeros of $(\Delta^{\mathrm{pos}})^{(1)}$ and Δ^{pos}, respectively.

5.2.2. Two-positon solution. Let $f_i := \cos\theta_i\exp(-t\cos\omega_i)\cos(n\omega_i - t\sin\omega_i)$, $g_i := \sin\theta_i\exp(-t\cos\omega_i)\sin(n\omega_i - t\sin\omega_i)$, $i = 1, 2$ be solutions of (17) w.r.t. $\lambda_i = 2\cos\omega_i$, $\theta_i \neq \frac{n}{2}\pi$ $(n \in \mathbb{Z})$. Define

$$h_i = f_i + g_i = \exp(-t\cos\omega_i)\cos(Y_i), \qquad Y_i = n\omega_i - t\sin\omega_i - \theta_i,$$
$$Z_i = \partial_{\omega_i}Y_i = n - t\cos\omega_i, \qquad \eta_i = Z_i + \tfrac{1}{2}\beta_i(t)\sin(2\theta_i).$$

Then using theorem 3 with $l = 0$, $I = 2$, $\mathbf{m} = (2, 2)$, one obtains the 2-positon solution for TLSCS (12),

$$x^{2\mathrm{p}} = \frac{1}{2}\log\left[\frac{\mathrm{cas}(\cos Y_1, \eta_1\sin Y_1, \cos Y_2, \eta_2\sin Y_2)}{\mathrm{cas}(\cos Y_1, \eta_1\sin Y_1, \cos Y_2, \eta_2\sin Y_2)^{(1)}}\right]^2,$$

$$\phi_{1+}^{2\mathrm{p}} = -\frac{1}{2}\sin(2\theta_1)\exp(-t\cos\omega_1)\frac{\mathrm{cas}(\sin Y_1, \cos Y_1, \eta_1\sin Y_1, \cos Y_2, \eta_2\sin Y_2)}{\mathrm{cas}(\cos Y_1, \eta_1\sin Y_1, \cos Y_2, \eta_2\sin Y_2)^{(1)}},$$

$$\phi_{1-}^{2\mathrm{p}} = -\dot{\beta}_1\exp(t\cos\omega_i)\frac{\mathrm{cas}(\cos Y_1, \cos Y_2, \eta_2\sin Y_2)^{(1)}}{\mathrm{cas}(\cos Y_1, \eta_1\sin Y_1, \cos Y_2, \eta_2\sin Y_2)},$$

and $\phi_{2\pm}^{2\mathrm{p}}$ have the similar formulae. $x^{2\mathrm{p}}$ describes that a wave profile oscillates, decreases to zero slowly as $|n| \to \infty$. $\phi_{1\pm}^{2\mathrm{p}}$ and $\phi_{2\pm}^{2\mathrm{p}}$ behave like $\mathcal{O}(n^{-1})$ as $|n| \to \infty$.

The positon–positon interaction can be analysed as follows. Fixing η_1, assume $|\eta_2| \to \infty$ when $t \to \pm\infty$, then

$$x^{2\mathrm{p}} \sim \frac{1}{2}\log\left[\frac{A\left(\eta_1+\frac{3}{2}\right)+B\sin(2Y_1+3\omega_1)}{A\left(\eta_1+\frac{5}{2}\right)+B\sin(2Y_1+5\omega_1)}\right]^2, \qquad \text{as} \quad t \to \pm\infty,$$

where

$$A := 2\sin\omega_1\sin\omega_2 + \sin(3\omega_1)\sin\omega_2 - 2\sin(2\omega_1)\sin(2\omega_2) + \sin\omega_1\sin(3\omega_2),$$
$$B := \sin\omega_2\cos(2\omega_1) + \tfrac{3}{2}\sin\omega_2 - \sin(2\omega_2)\cos\omega_1 + \tfrac{1}{2}\sin(3\omega_2).$$

Thus, we observe one-positon in terms of η_1 and Y_1 at $t = \pm\infty$. And there is no phase shift during the interaction. Analogously, fixing η_2 we observe one-positon at $t = \pm\infty$ in terms of η_2 and Y_2. Similarly, there is no phase shift in the course of interaction. Positons propagate transparently as if others were absent. This transparency of interaction is a remarkable feature of positon solutions. The N-positon solutions are obtained by taking eigenfunctions $f_i := \cos\theta_i \exp(-t\cos\omega_i)\cos(n\omega_i - t\sin\omega_i)$, $g_i := \sin\theta_i \exp(-t\cos\omega_i)\sin(n\omega_i - t\sin\omega_i)$ of (17) w.r.t. $\lambda_i = 2\cos\omega_i, \theta_i \neq \frac{n}{2}\pi\ (n \in \mathbb{Z}), i = 1, \ldots, N$, $h_i = f_i + g_i$ and by using theorem 3 with $l = 0, I = N, \mathbf{m} = (2, 2, \ldots, 2) \in \mathbb{N}^N$.

We can construct N-negaton solutions to TLSCS (12) by considering eigenfunctions $f_i := \epsilon_i^n \exp(n\omega_i - \epsilon_i\,\mathrm{e}^{\omega_i} t + \theta_i)$, $g_i := \epsilon_i^n \exp(-n\omega_i - \epsilon_i\,\mathrm{e}^{-\omega_i} t - \theta_i)$ of (17) with $\lambda_i = 2\epsilon_i\cosh\omega_i$, distinct $\omega_i \in \mathbb{R}, \theta_i \in \mathbb{R}, \epsilon_i = \pm 1, h_i = f_i + g_i$ and using theorem 3 with $l = 0, I = N, \mathbf{m} = (2, 2, \cdots, 2) \in \mathbb{N}^N$.

The GFDT is quite general to construct not only N-soliton (positon, negaton) solutions, but also the solutions of combined type.

5.3. *Positon–negaton solution and interaction*

Let

$$f_1 := \cos\theta_1 \exp(-t\cos\omega_1)\cos(n\omega_1 - t\sin\omega_1),$$
$$g_1 := \sin\theta_1 \exp(-t\cos\omega_1)\sin(n\omega_1 - t\sin\omega_1),$$
$$f_2 := \epsilon_2^n \exp(n\omega_2 - \epsilon_2\,\mathrm{e}^{\omega_2} t + \theta_2),$$
$$g_2 := \epsilon_2^n \exp(-n\omega_2 - \epsilon_2\,\mathrm{e}^{-\omega_2} t - \theta_2),$$

where $\theta_1, \theta_2, \omega_1$ and ω_2 are real numbers, $\theta_1 \neq k\pi/2$ for any integer k. Define

$$h_1 = f_1 + g_1 = \exp(-t\cos\omega_1)\cos Y_1, \; h_2 = f_2 + g_2 = 2\epsilon_2^n \exp(-t\epsilon_2\cosh\omega_2)\cosh Y_2,$$

where

$$Y_1 := n\omega_1 - t\sin\omega_1 - \theta_1, \qquad Y_2 := n\omega_2 - t\epsilon_2\sinh\omega_2 + \theta_2.$$

Denote

$$\eta_1 := \partial_{\omega_1} Y_1 + \tfrac{1}{2}\beta_1(t)\sin 2\theta_1, \qquad \eta_2 := \partial_{\omega_2} Y_2 + \tfrac{1}{2}\beta_2(t).$$

Then according to theorem 3 with the specification $l = 0, I = 2, \mathbf{m} = (2, 2)$, we obtain the positon–negaton solution for TLSCS (12).

To analyse the interaction, we always assume that $\beta_1(t)$ and $\beta_2(t)$ tend to $\mp\infty$ as $n \to \pm\infty$, and all parameters, including $\omega_1\omega_2$ and ϵ_2, are positive. Fixing η_1, if for particular $\beta_2(t)$, η_2 increases faster than $\sinh(2Y_2)$ (i.e. $|\eta_2/\sinh(2Y_2)| \to \infty$) as $t \to \infty$, which indicates that negaton travels at a very high speed dominated by $\beta_2(t)$ then

$$x^{\mathrm{pm}} \sim \frac{1}{2}\log\left[\frac{\frac{1}{2}\sin(2Y_1 + 3\omega_1) + \sin\omega_1(\eta_1 - \Delta_8)}{\frac{1}{2}\sin(2Y_1 + 5\omega_1) + \sin\omega_1(\eta_1 + 1 - \Delta_8)}\right], \qquad \text{as} \quad t \to \pm\infty,$$

where

$$\Delta_8 := \sin\omega_1 \frac{\cos\omega_1 - 2\epsilon_2\cosh\omega_2 - 2\cosh\omega_2\cos\omega_1 + 2}{\mathrm{e}^{\omega_2}(\epsilon_2\,\mathrm{e}^{-\omega_2} - \mathrm{e}^{\mathrm{i}\omega_1})(\epsilon_2\,\mathrm{e}^{-\omega_2} - \mathrm{e}^{-\mathrm{i}\omega_1})}.$$

Thus we see the one-positon profile (see (22*c*)) with neither phase shift nor displacement at the two end. That is to say the negaton is transparent for positon, which is a phenomenon *never* observed in the ordinary Toda lattice case.

If η_2 increases slower than $\sinh(2Y_2)$ (i.e. $|\eta_2/\sinh(2Y_2)| \to \infty$) as $|t|$ increases, which implies a slowly travelling negaton in comparison with the previous case, then

$$x^{pn} \sim \frac{1}{2}\log\left[\frac{\sin\omega_1(\eta_1+\Delta_9)+\frac{1}{2}\sin(2Y_1+\omega_1-\Delta_{11})}{\sin\omega_1(\eta_1+1+\Delta_9)+\frac{1}{2}\sin(2Y_1+3\omega_1-\Delta_{11})}\right]^2 - 2\omega_2, \qquad \text{as} \quad t\to-\infty,$$

$$x^{pn} \sim \frac{1}{2}\log\left[\frac{\sin\omega_1(\eta_1+\Delta_{10})+\frac{1}{2}\sin(2Y_1+5\omega_1+\Delta_{11})}{\sin\omega_1(\eta_1+1+\Delta_{10})+\frac{1}{2}\sin(2Y_1+7\omega_1+\Delta_{11})}\right]^2 + 2\omega_2, \qquad \text{as} \quad t\to+\infty,$$

where

$$\Delta_9 := \frac{1}{2}\,\mathrm{e}^{-2\omega_2}\frac{(\epsilon_2\,\mathrm{e}^{\omega_2}-3\mathrm{e}^{\mathrm{i}\omega_1})(\epsilon_2\,\mathrm{e}^{\omega_2}-3\,\mathrm{e}^{-\mathrm{i}\omega_1})-4}{(\epsilon_2\,\mathrm{e}^{-\omega_2}-\mathrm{e}^{\mathrm{i}\omega_1})(\epsilon_2\,\mathrm{e}^{-\omega_2}-\mathrm{e}^{-\mathrm{i}\omega_1})},$$

$$\Delta_{10} := \frac{1}{2}\,\mathrm{e}^{2\omega_2}\frac{(\epsilon_2\,\mathrm{e}^{-\omega_2}-3\,\mathrm{e}^{\mathrm{i}\omega_1})(\epsilon_2\,\mathrm{e}^{-\omega_2}-3\,\mathrm{e}^{-\mathrm{i}\omega_1})-4}{(\epsilon_2\,\mathrm{e}^{\omega_2}-\mathrm{e}^{\mathrm{i}\omega_1})(\epsilon_2\,\mathrm{e}^{\omega_2}-\mathrm{e}^{-\mathrm{i}\omega_1})},$$

and

$$\Delta_{11} := \mathrm{i}\log\left(\frac{\epsilon_2\,\mathrm{e}^{\omega_2}-\mathrm{e}^{\mathrm{i}\omega_1}}{\epsilon_2\,\mathrm{e}^{\omega_2}-\mathrm{e}^{-\mathrm{i}\omega_1}}\right)^2$$

are all real constants. Thus positon travels with the phase shift determined by Δ_9, Δ_{10} and Δ_{11}, with the displacement determined by $4\omega_2$ in the course of collision. This is general phenomenon caused by the existence of negaton.

If we fix a coordinate frame which travelling with negaton profile, then

$$x^{pn} \sim \frac{1}{2}\log\left[\frac{\frac{1}{2}\sinh(2Y_2+3\omega_2)+(\eta_2+3)\sinh\omega_2}{\frac{1}{2}\sinh(2Y_2+5\omega_2)+(\eta_2+4)\sinh\omega_2}\right]^2 \qquad \text{as} \quad t\to\pm\infty.$$

Thus, the negaton travels insensitive about the existence of positon.

Soliton–positon and soliton–negaton solutions can obtained in the same way by using theorem 3 with $l = 1, I = 1, \mathbf{m} = 2, q := F+\alpha G, h := f_1+g_1$ and $q := F+\alpha G$, $h := f_2+g_2$ respectively. ($f_i, g_i i = 1, 2$ are defined in the beginning of this section and F, G, α are defined in section 5.1.)

6. Conclusions

On the basis of the constrained flows of Toda lattice hierarchy, we constructed Toda lattice hierarchy with self-consistent sources and their Lax representation.

We developed a method to construct FDT with arbitrary functions of time and the GFDT with arbitrary functions of time, which, in contrast with the well-known Darboux transformation for Toda lattice, provide the non-auto-Bäcklund transformation between two TLSCSs with different degrees and enable us to obtain various explicit solutions to TLSCS. Resembling the ordinary Toda lattice case, this system possesses the solutions of rich families, including solitons, positons, negatons and the solutions of combined types. A number of solutions are listed by our method. The investigation on these solutions shows a quite similar nature with solutions of the ordinary Toda lattice. However, the new feature concerning interactions between negaton and positon (or soliton) which is different from the ordinary Toda lattice case is also stated. This difference is caused by the wide range of variation of the speed of negatons in the TLSCS cases. We note that the variation of speed is a common feature for continuous and discrete systems with self-consistent sources, see [6, 7] etc.

It is convinced that our approach for constructing systems with self-consistent sources and generalized forward Darboux transformation technique are available for other discrete systems. Some investigation will be present in the forthcoming paper.

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